Introduction of Vector

Physical quantities having magnitude, direction and obeying laws of vector algebra are called vectors.

Example : Displacement, velocity, acceleration, momentum, force, impulse, weight, thrust, torque, angular momentum, angular velocity *etc*.

If a physical quantity has magnitude and direction both, then it does not always imply that it is a vector. For it to be a vector the third condition of obeying laws of vector algebra has to be satisfied.

Example : The physical quantity current has both magnitude and direction but is still a scalar as it disobeys the laws of vector algebra.

Types of Vector

(1) **Equal vectors** : Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal when they have equal magnitudes and same direction.

(2) **Parallel vector** : Two vectors $\vec{A}$ and $\vec{B}$ are said to be parallel when

(i) Both have same direction.

(ii) One vector is scalar (positive) non-zero multiple of another vector.

(3) **Anti-parallel vectors** : Two vectors $\vec{A}$ and $\vec{B}$ are said to be anti-parallel when

(i) Both have opposite direction.

(ii) One vector is scalar non-zero negative multiple of another vector.

(4) **Collinear vectors** : When the vectors under consideration can share the same support or have a common support then the considered vectors are collinear.

(5) **Zero vector** $(\vec{0})$: A vector having zero magnitude and arbitrary direction (not known to us) is a zero vector.

(6) **Unit vector** : A vector divided by its magnitude is a unit vector. Unit vector for $\vec{A}$ is $\hat{A}$ (read as A cap or A hat).

Since, $\hat{A} = \dfrac{\vec{A}}{A} \Rightarrow \vec{A} = A\hat{A}$.

Thus, we can say that unit vector gives us the direction.

(7) **Orthogonal unit vectors** $\hat{i}, \hat{j}$ and $\hat{k}$ are called orthogonal unit vectors. These vectors must form a Right Handed Triad (It is a coordinate system such that when we Curl the fingers of right hand from x to y then we must get the direction of z along thumb). The

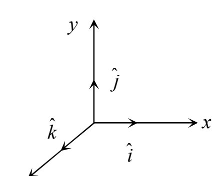


Fig. 0.1

$$\hat{i} = \frac{\vec{x}}{x}, \hat{j} = \frac{\vec{y}}{y}, \hat{k} = \frac{\vec{z}}{z}$$

$$\therefore \vec{x} = x\hat{i}, \ \vec{y} = y\hat{j}, \ \vec{z} = z\hat{k}$$

(8) **Polar vectors** : These have starting point or point of application . Example displacement and force *etc*.

(9) **Axial Vectors** : These represent rotational effects and are always along the axis of rotation in accordance with right hand screw rule. Angular velocity, torque and angular momentum, etc., are example of physical quantities of this type.

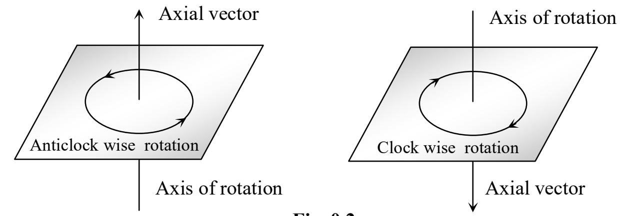


Fig. 0.2

(10) **Coplanar vector** : Three (or more) vectors are called coplanar vector if they lie in the same plane. Two (free) vectors are always coplanar.

Triangle Law of Vector Addition of Two Vectors

If two non zero vectors are represented by the two sides of a triangle taken in same order then the resultant is given by the closing side of triangle in opposite order. *i.e.* $\vec{R} = \vec{A} + \vec{B}$

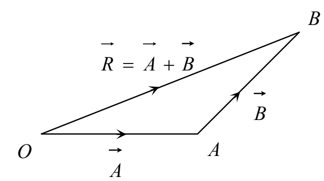


Fig. 0.3

$$\because \ \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

(1) **Magnitude of resultant vector**

In ΔABN, $\cos\theta = \dfrac{AN}{B} \ \therefore \ AN = B\cos\theta$

$\sin\theta = \dfrac{BN}{B} \ \therefore \ BN = B\sin\theta$

In ΔOBN, we have $OB^2 = ON^2 + BN^2$

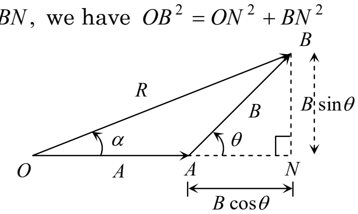


Fig. 0.4

$$\Rightarrow R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

$$\Rightarrow R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\Rightarrow R^2 = A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$$

$$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

(2) **Direction of resultant vectors** : If θ is angle between $\vec{A}$ and $\vec{B}$, then

$| \vec{A} + \vec{B} | = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

If $\vec{R}$ makes an angle α with $\vec{A}$, then in ΔOBN,

$$\tan\alpha = \frac{BN}{ON} = \frac{BN}{OA + AN}$$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

Parallelogram Law of Vector Addition

If two non zero vectors are represented by the two adjacent sides of a parallelogram then the resultant is given by the diagonal of the parallelogram passing through the point of intersection of the two vectors.

(1) **Magnitude**

Since, $R^2 = ON^2 + CN^2$

$\Rightarrow R^2 = (OA + AN)^2 + CN^2$

$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$

$\therefore \quad R = |\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

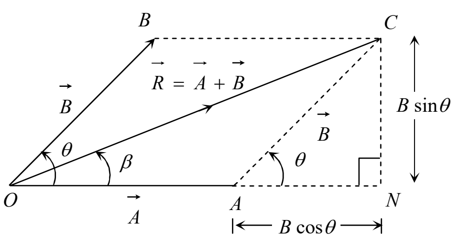


Fig. 0.5

Special cases: $R = A + B$ when $\theta = 0^o$

$R = A - B$ when $\theta = 180^o$

$R = \sqrt{A^2 + B^2}$ when $\theta = 90^o$

(2) **Direction**

$$\tan\beta = \frac{CN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}$$

Polygon Law of Vector Addition

If a number of non zero vectors are represented by the $(n-1)$ sides of an n-sided polygon then the resultant is given by the closing side or the n^{th} side of the polygon taken in opposite order. So,

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$$

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{OE}$$

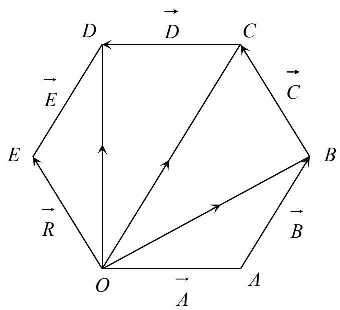


Fig. 0.6

Note : ❑ Resultant of two unequal vectors can not be zero.

❑ Resultant of three co-planar vectors may or may not be zero

❑ Resultant of three non co- planar vectors can not be zero.

Subtraction of vectors

Since, $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ and

$| \vec{A} + \vec{B} | = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow | \vec{A} - \vec{B} | = \sqrt{A^2 + B^2 + 2AB\cos(180^o - \theta)}$

Since, $\cos(180 - \theta) = -\cos\theta$

$\Rightarrow | \vec{A} - \vec{B} | = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

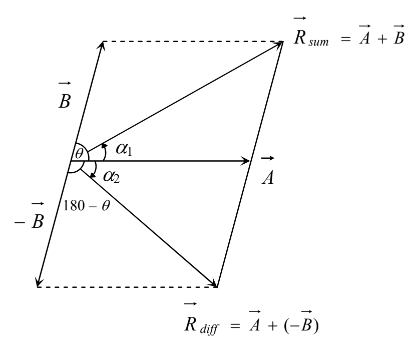


Fig. 0.7

$$\tan\alpha_1 = \frac{B\sin\theta}{A + B\cos\theta}$$

and $\tan\alpha_2 = \dfrac{B\sin(180 - \theta)}{A + B\cos(180 - \theta)}$

But $\sin(180 - \theta) = \sin\theta$ and $\cos(180 - \theta) = -\cos\theta$

$$\Rightarrow \tan\alpha_2 = \frac{B\sin\theta}{A - B\cos\theta}$$

Resolution of Vector Into Components

Consider a vector $\vec{R}$ in X-Y plane as shown in fig. If we draw orthogonal vectors $\vec{R}_x$ and $\vec{R}_y$ along x and y axes respectively, by law of vector addition, $\vec{R} = \vec{R}_x + \vec{R}_y$

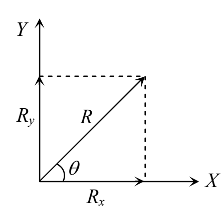


Fig. 0.8

Now as for any vector $\vec{A} = A\hat{n}$

so, $\vec{R}_x = \hat{i}R_x$ and $\vec{R}_y = \hat{j}R_y$

so $\vec{R} = \hat{i}R_x + \hat{j}R_y$ ...(i)

But from figure $R_x = R\cos\theta$ ...(ii)

and $R_y = R\sin\theta$ ...(iii)

Since R and θ are usually known, Equation (ii) and (iii) give the magnitude of the components of $\vec{R}$ along x and y-axes respectively.

Here it is worthy to note once a vector is resolved into its components, the components themselves can be used to specify the vector as

(1) The magnitude of the vector $\vec{R}$ is obtained by squaring and adding equation (ii) and (iii), *i.e.*

$$R = \sqrt{R_x^2 + R_y^2}$$

(2) The direction of the vector $\vec{R}$ is obtained by dividing equation (iii) by (ii), *i.e.*

$$\tan\theta = (R_y / R_x) \text{ or } \theta = \tan^{-1}(R_y / R_x)$$

Rectangular Components of 3-D Vector

$$\vec{R} = \vec{R}_x + \vec{R}_y + \vec{R}_z q \text{ or } \vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

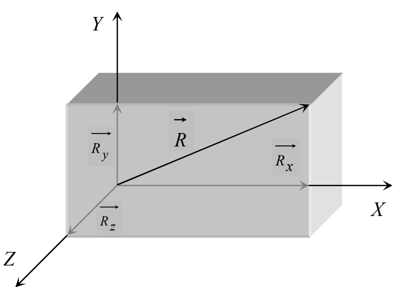


Fig. 0.9

If $\vec{R}$ makes an angle α with x axis, β with y axis and γ with z axis, then

$$\Rightarrow \cos\alpha = \frac{R_x}{R} = \frac{R_x}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = l$$

$$\Rightarrow \cos\beta = \frac{R_y}{R} = \frac{R_y}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = m$$

$$\Rightarrow \cos\gamma = \frac{R_z}{R} = \frac{R_z}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = n$$

Where *l, m, n* are called Direction Cosines of the vector $\vec{R}$ and

$$l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{R_x^2 + R_y^2 + R_z^2}{R_x^2 + R_y^2 + R_z^2} = 1$$

Note : ❑ When a point P have coordinate (x, y, z) then its position vector $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

❑ When a particle moves from point (x_1, y_1, z_1) to (x_2, y_2, z_2) then its displacement vector

$$\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Scalar Product of Two Vectors

(1) **Definition :** The scalar product (or dot product) of two vectors is defined as the product of the magnitude of two vectors with cosine of angle between them.

Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having angle θ between them, then their scalar product written as $\vec{A}.\vec{B}$ is defined as $\vec{A}.\vec{B} = AB\cos\theta$

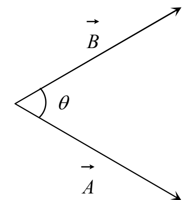


Fig. 0.10

(2) **Properties :** (i) It is always a scalar which is positive if angle between the vectors is acute (*i.e.*, < 90°) and negative if angle between them is obtuse (*i.e.* 90°< θ < 180°).

(ii) It is commutative, *i.e.* $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

(iii) It is distributive, *i.e.* $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

(iv) As by definition $\vec{A}.\vec{B} = AB\cos\theta$

The angle between the vectors $\theta = \cos^{-1}\left[\frac{\vec{A}.\vec{B}}{AB}\right]$

(v) Scalar product of two vectors will be maximum when $\cos\theta = \max = 1$, *i.e.* $\theta = 0^o$, *i.e.*, vectors are parallel

$$(\vec{A}.\vec{B})_{\max} = AB$$

(vi) Scalar product of two vectors will be minimum when $|\cos\theta| = \min = 0$, *i.e.* $\theta = 90^o$

$$(\vec{A}.\vec{B})_{\min} = 0$$

i.e. if the scalar product of two nonzero vectors vanishes the vectors are orthogonal.

(vii) The scalar product of a vector by itself is termed as self dot product and is given by $(\vec{A})^2 = \vec{A}.\vec{A} = AA\cos\theta = A^2$

i.e. $A = \sqrt{\vec{A}.\vec{A}}$

(viii) In case of unit vector $\hat{n}$

$\hat{n}.\hat{n} = 1 \times 1 \times \cos 0 = 1$ so $\hat{n}.\hat{n} = \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

(ix) In case of orthogonal unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 1 \times 1 \cos 90° = 0$

(x) In terms of components

$$\vec{A}.\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_ZB_z]$$

(3) **Example** : (i) Work W: In physics for constant force work is defined as, $W = Fs\cos\theta$ ...(i)

But by definition of scalar product of two vectors, $\vec{F}.\vec{s} = Fs\cos\theta$ ...(ii)

So from eq^n (i) and (ii) $W = \vec{F}.\vec{s}$ *i.e.* work is the scalar product of force with displacement.

(ii) Power P:

As $W = \vec{F}.\vec{s}$ or $\frac{dW}{dt} = \vec{F}.\frac{d\vec{s}}{dt}$ [As $\vec{F}$ is constant]

or $P = \vec{F}.\vec{v}$ *i.e.*, power is the scalar product of force with velocity.

$\left[\text{As} \frac{dW}{dt} = P \text{ and } \frac{d\vec{s}}{dt} = \vec{v}\right]$

(iii) Magnetic Flux ϕ:

Magnetic flux through an area is given by $d\phi = B\,ds\cos\theta$ ...(i)

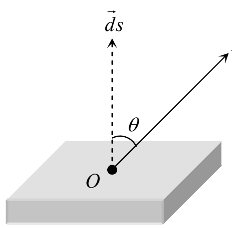


Fig. 0.11

But by definition of scalar product $\vec{B}.d\vec{s} = Bds\cos\theta$ ...(ii)

So from eq^n (i) and (ii) we have

$d\phi = \vec{B}.d\vec{s}$ or $\phi = \int \vec{B}.d\vec{s}$

(iv) Potential energy of a dipole U: If an electric dipole of moment $\vec{p}$ is situated in an electric field $\vec{E}$ or a magnetic dipole of moment $\vec{M}$ in a field of induction $\vec{B}$, the potential energy of the dipole is given by :

$U_E = -\vec{p}.\vec{E}$ and $U_B = -\vec{M}.\vec{B}$

Vector Product of Two Vectors

(1) **Definition** : The vector product or cross product of two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them, and direction perpendicular to the plane containing the two vectors in accordance with right hand screw rule.

$$\vec{C} = \vec{A} \times \vec{B}$$

Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product written as $\vec{A} \times \vec{B}$ is a vector $\vec{C}$ defined by

$$\vec{C} = \vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$$

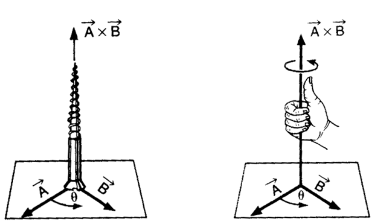


Fig. 0.12

The direction of $\vec{A} \times \vec{B}$, *i.e.* $\vec{C}$ is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and in the sense of advance of a right handed screw rotated from $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus, if a right handed screw whose axis is perpendicular to the plane framed by $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A} \times \vec{B}$ *i.e.* $\vec{C}$

(2) **Properties**

(i) Vector product of any two vectors is always a vector perpendicular to the plane containing these two vectors, *i.e.*, orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

(ii) Vector product of two vectors is not commutative, *i.e.*, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ [but $= -\vec{B} \times \vec{A}$]

Here it is worthy to note that

$|\vec{A} \times \vec{B}| = |\vec{B} \times \vec{A}| = AB\sin\theta$

i.e. in case of vector $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ magnitudes are equal but directions are opposite.

(iii) The vector product is distributive when the order of the vectors is strictly maintained, *i.e.*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iv) The vector product of two vectors will be maximum when $\sin\theta = \max = 1$, *i.e.*, $\theta = 90^o$

$[\vec{A} \times \vec{B}]_{\max} = AB\,\hat{n}$

i.e. vector product is maximum if the vectors are orthogonal.

(v) The vector product of two non- zero vectors will be minimum when $|\sin\theta| =$ minimum = 0, *i.e.*, $\theta = 0^o$ or 180^o

$[\vec{A} \times \vec{B}]_{\min} = 0$

i.e. if the vector product of two non-zero vectors vanishes, the vectors are collinear.

(vi) The self cross product, *i.e.*, product of a vector by itself vanishes, *i.e.*, is null vector $\vec{A} \times \vec{A} = AA \sin 0^o \, \hat{n} = \vec{0}$

(vii) In case of unit vector $\hat{n} \times \hat{n} = \vec{0}$ so that $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

(viii) In case of orthogonal unit vectors, $\hat{i}, \hat{j}, \hat{k}$ in accordance with right hand screw rule :

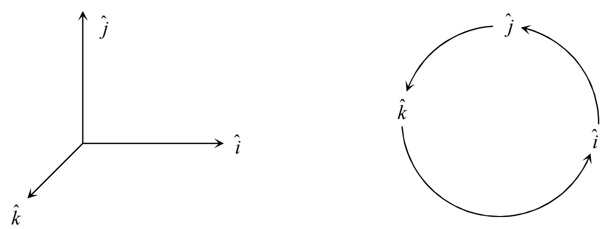


Fig. 0.13

$\hat{i} \times \hat{j} = \hat{k}, \ \hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$

And as cross product is not commutative,

$\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$ and $\hat{i} \times \hat{k} = -\hat{j}$

(x) In terms of components

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$$

(3) **Example** : Since vector product of two vectors is a vector, vector physical quantities (particularly representing rotational effects) like torque, angular momentum, velocity and force on a moving charge in a magnetic field and can be expressed as the vector product of two vectors. It is well – established in physics that :

(i) Torque $\vec{\tau} = \vec{r} \times \vec{F}$

(ii) Angular momentum $\vec{L} = \vec{r} \times \vec{p}$

(iii) Velocity $\vec{v} = \vec{\omega} \times \vec{r}$

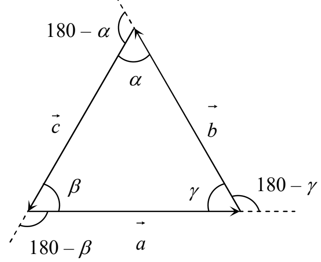
(iv) Force on a charged particle q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by $\vec{F} = q(\vec{v} \times \vec{B})$

(v) Torque on a dipole in a field $\vec{\tau_E} = \vec{p} \times \vec{E}$ and $\vec{\tau_B} = \vec{M} \times \vec{B}$

Lami's Theorem

In any ΔABC with sides $\vec{a}, \vec{b}, \vec{c}$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$



i.e. for any triangle the ratio of the sine of the angle containing the side to the length of the side is a constant.

For a triangle whose three sides are in the same order we establish the Lami's theorem in the following manner. For the triangle shown

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$ [All three sides are taken in order] ...(i)

$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$ ...(ii)

Pre-multiplying both sides by $\vec{a}$

$\vec{a} \times (\vec{a} + \vec{b}) = -\vec{a} \times \vec{c} \Rightarrow \vec{0} + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(iii)

Pre-multiplying both sides of (ii) by $\vec{b}$

$\vec{b} \times (\vec{a} + \vec{b}) = -\vec{b} \times \vec{c} \Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c}$

$\Rightarrow -\vec{a} \times \vec{b} = -\vec{b} \times \vec{c} \Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ...(iv)

From (iii) and (iv), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Taking magnitude, we get $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

$\Rightarrow ab \sin(180 - \gamma) = bc \sin(180 - \alpha) = ca \sin(180 - \beta)$

$\Rightarrow ab \sin\gamma = bc \sin\alpha = ca \sin\beta$

Dividing through out by *abc,* we have

$$\Rightarrow \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$